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Adding and Subtracting

Numbers can be treated as though they have two parts: a positive or negative sign, and a number. Numbers without any sign are understood to be positive.

To add two numbers that have the same sign, add the number parts and keep the sign. Example: To add \((-6) + (-3)\), add 6 and 3, and then attach the negative sign from the original numbers to the sum. \((-6) + (-3) = -9\).

To add two numbers that have different signs, find the difference between the number parts and keep the sign of the number whose number part is larger. Example: To add \((-7) + (+4)\), subtract 4 from 7 to get 3. \(7 > 4\) (the number part of \(-7\) is greater than the number part of 4), so the final sum will be negative: \((-7) + (+4) = -3\).

Subtraction is the opposite of addition. You can rephrase any subtraction problem as an addition problem by changing the operation sign from a minus to a plus and switching the sign on the second number. Example: \(8 - 5 = 8 + (-5)\). There's no real advantage to rephrasing if you are subtracting a smaller positive number from a larger positive number. But the concept comes in very handy when you are subtracting a negative number from any other number, a positive number from a negative number, or a larger positive number from a smaller positive number.

To subtract a negative number, rephrase as an addition problem and follow the rules for addition of signed numbers. For instance, \(9 - (-10) = 9 + 10 = 19\).

To subtract a positive number from a negative number, or from a smaller positive number, change the sign of the number that you are subtracting from positive to negative and follow the rules for addition of signed numbers.

Example:

\[ (-4) - 1 = (-4) + (-1) = -5 \]
Multiplication and Division of Signed Numbers

Multiplying or dividing two numbers with the same sign gives a positive result.

Examples:

\((-4)(-7) = +28\)
\((-50) \div (-5) = +10\)

Multiplying or dividing two numbers with different signs gives a negative result.

Examples:

\((-2)(+3) = -6\)
\(8 \div (-4) = -2\)
Absolute Value

The absolute value of a number is the value of a number without its sign. It is written as two vertical lines, one on either side of the number and its sign.

Example:

\[ | -3| = |3| = 3 \]

The absolute value of a number can be thought of as the number's distance from zero on the number line. Since both 3 and \(-3\) are 3 units from 0, each has an absolute value of 3. If you are told that \( |x| = 5 \), \( x \) could equal 5 or \(-5\).
Properties of Zero

Adding zero to or subtracting zero from a number does not change the number.

\[ x + 0 = x \]
\[ 0 + x = x \]
\[ x - 0 = x \]

**Examples:**

\[ 5 + 0 = 5 \]
\[ 0 + (-3) = -3 \]
\[ 4 - 0 = 4 \]

Notice, however, that subtracting a number from zero changes the number’s sign. It’s easy to see why if you rephrase the problem as an addition problem.

**Example:**

Subtract 5 from 0.

\[ 0 - 5 = -5 \] That's because \[ 0 - 5 = 0 + (-5) \], and according to the rules for addition with signed numbers, \[ 0 + (-5) = -5 \].
The product of zero and any number is zero.

**Examples:**

\[(0)(z) = 0\]
\[(z)(0) = 0\]
\[(0)(12) = 0\]

Division by zero is undefined. For GMAT purposes, that translates as “it can’t be done.” Since fractions are essentially division (that is, \[\frac{1}{4}\] means \[1 \div 4\]), any fraction with zero in the denominator is also undefined. So when you are given a fraction that has an algebraic expression in the denominator, be sure that the expression cannot equal zero.
Properties of 1 and \(-1\)

Multiplying or dividing a number by 1 does not change the number.

\[(a)(1) = a\]
\[(1)(a) = a\]
\[a \div 1 = a\]

**Examples:**

\[(4)(1) = 4\]
\[(1)(-5) = -5\]
\[(-7) \div 1 = -7\]

Multiplying or dividing a nonzero number by \(-1\) changes the sign of the number.

\[(a)(-1) = -a\]
\[(-1)(a) = -a\]
\[a \div (-1) = -a\]

**Examples:**

\[(6)(-1) = -6\]
\[(-3)(-1) = 3\]
\[(-8) \div (-1) = 8\]
Factors, Multiples, and Remainders

Multiples and Divisibility

A multiple is the product of a specified number and an integer. For example, 3, 12, and 90 are all multiples of 3. \(3 = (3)(1);\) \(12 = (3)(4);\) and \(90 = (3)(30).\) 4 is not a multiple of 3, because there is no integer that can be multiplied by 3 and yield 4.

The concepts of multiples and factors are tied together by the idea of divisibility. A number is said to be evenly divisible by another number if the result of the division is an integer with no remainder. A number that is evenly divisible by a second number is also a multiple of the second number.

For example, \(52 \div 4 = 13,\) which is an integer. So 52 is evenly divisible by 4, and it’s also a multiple of 4.

On some GMAT math problems, you will find yourself trying to assess whether one number is evenly divisible by another. You can use several simple rules to save time.

- An integer is divisible by 2 if its last digit is divisible by 2.
- An integer is divisible by 3 if its digits add up to a multiple of 3.
- An integer is divisible by 4 if its last two digits are a multiple of 4.
- An integer is divisible by 5 if its last digit is 0 or 5.
- An integer is divisible by 6 if it is divisible by 2 and 3.
- An integer is divisible by 9 if its digits add up to a multiple of 9.
Example:

6,930 is a multiple of 2, since 0 is even.

... a multiple of 3, since \(6 + 9 + 3 + 0 = 18\), which is a multiple of 3.

... not a multiple of 4, since 30 is not a multiple of 4.

... a multiple of 5, since it ends in zero.

... a multiple of 6, since it is a multiple of 2 and 3.

... a multiple of 9, since \(6 + 9 + 3 + 0 = 18\), a multiple of 9.

Properties of Odd/Even Numbers

*Even* numbers are integers that are evenly divisible by 2; *odd* numbers are integers that are not evenly divisible by 2. Integers whose last digit is 0, 2, 4, 6, or 8 are even; integers whose last digit is 1, 3, 5, 7, or 9 are odd. The terms *odd* and *even* apply only to integers, but may be used for either positive or negative integers. 0 is considered even.

Rules for Odds and Evens

- Odd + Odd = Even
- Even + Even = Even
- Odd + Even = Odd
- Odd × Odd = Odd
- Even × Even = Even
- Odd × Even = Even

Note that multiplying any even number by *any* integer always produces another even number.
It may be easier to pick numbers in problems that ask you to decide whether some unknown will be odd or even.

**Example:**

Is the sum of two odd numbers odd or even?

Pick any two odd numbers, for example, 3 and 5. \(3 + 5 = 8\). Since the sum of the two odd numbers that you picked is an even number, 8, it’s safe to say that the sum of any two odd numbers is even.

Picking Numbers will work in any odds/evens problem, no matter how complicated. The only time you have to be careful is when division is involved, especially if the problem is in Data Sufficiency format; different numbers may yield different results.

**Example:**

Integer \(x\) is evenly divisible by 2. Is \(\frac{x}{2}\) even?

By definition, any multiple of 2 is even, so integer \(x\) is even. And \(\frac{x}{2}\) must be an integer. But is \(\frac{x}{2}\) even or odd? In this case, picking two different even numbers for \(x\) can yield two different results. If you let \(x = 4\), then \(\frac{x}{2} = \frac{4}{2} = 2\) which is even. But if you let \(x = 6\), then \(\frac{x}{2} = \frac{6}{2} = 3\), which is odd. So \(\frac{x}{2}\) could be even or odd—and you wouldn’t know that if you picked only one number.
Factors and Primes

The factors, or divisors, of an integer are the positive integers by which it is evenly divisible (leaving no remainder).

Example:

What are the factors of 36?

36 has nine factors: 1, 2, 3, 4, 6, 9, 12, 18, and 36. We can group these factors in pairs: (1)(36) = (2)(18) = (3)(12) = (4)(9) = (6)(6)

The greatest common factor, or greatest common divisor, of a pair of integers is the largest factor that they share.

To find the greatest common factor, break down both integers into their prime factorizations and multiply all the prime factors they have in common. 36 = (2)(2)(3)(3), and 48 = (2)(2)(2)(2)(3). What they have in common is two 2's and one 3, so the GCF is (2)(2)(3) = 12.

A prime number is an integer greater than 1 that has only two factors: itself and 1. The number 1 is not considered a prime, because it is divisible only by itself. The number 2 is the smallest prime number and the only even prime. (Any other even number must have 2 as a factor, and therefore cannot be prime.)

Prime Factors

The prime factorization of a number is the expression of the number as the product of its prime factors (the factors that are prime numbers).
There are two common ways to determine a number's prime factorization. The rules given above for determining divisibility by certain numbers come in handy in both methods.

**Method #1:** Work your way up through the prime numbers, starting with 2. (You'll save time in this process, especially when you're starting with a large number, by knowing the first ten prime numbers by heart: 2, 3, 5, 7, 11, 13, 17, 19, 23, and 29.)

**Example:**

What is the prime factorization of 210?

\[ 210 = (2)(105) \]

Since 105 is odd, it can't contain another factor of 2. The next smallest prime number is 3. The digits of 105 add up to 6, which is a multiple of 3, so 3 is a factor of 105.

\[ 210 = (2)(3)(35) \]

The digits of 35 add up to 8, which is not a multiple of 3. But 35 ends in 5, so it is a multiple of the next largest prime number, 5.

\[ 210 = (2)(3)(5)(7) \]

Since 7 is a prime number, this equation expresses the complete prime factorization of 210.

**Method #2:** Figure out one pair of factors, and then determine their factors, continuing the process until you're left with only prime numbers. Those primes will be the prime factorization.
Example:

What is the prime factorization of 1,050?

The discrete prime factors of 1,050 are therefore 2, 5, 3, and 7, with the prime number 5 occurring twice in the prime factorization. We usually write out the prime factorization by putting the prime numbers in increasing order. Here, that would be \(2 \cdot 3 \cdot 5 \cdot 5 \cdot 7\). The prime factorization can also be expressed in exponential form: \(2 \cdot 3 \cdot 5^2 \cdot 7\).

The Least Common Multiple

The least common multiple of two or more integers is the smallest number that is a multiple of each of the integers. Here’s one quick way to find it:

1. Determine the prime factorization of each integer.
2. Write out each prime number the maximum number of times that it appears in any one of the prime factorizations.
3. Multiply those prime numbers together to get the least common multiple of the original integers.
Example:
What is the least common multiple of 6 and 8?

Start by finding the prime factors of 6 and 8.

\[ 6 = (2)(3) \]
\[ 8 = (2)(2)(2) \]

2 appears as a factor three times in the prime factorization of 8, while 3 appears as only a single factor of 6. So the least common multiple of 6 and 8 will be \((2)(2)(2)(3)\), or 24.

Note that the least common multiple of two integers is smaller than their product if they have any factors in common. For instance, the product of 6 and 8 is 48, but their least common multiple is only 24.

Although you won’t see the term least common multiple very often on the GMAT, you’ll find the concept useful whenever you’re adding or subtracting fractions with different denominators.

Remainders

The remainder is what is “left over” in a division problem. A remainder is always smaller than the number you are dividing by. For instance, 17 divided by 3 is 5, with a remainder of 2. 12 divided by 6 is 2, with a remainder of 0 (since 12 is evenly divisible by 6).

GMAT writers often disguise remainder problems. For instance, a problem might state that the slats of a fence are painted in three colors, which appear in a fixed order, such as red, yellow, blue, red, yellow, blue. You would then be asked something like, “If the first slat is red, what color is the 301st slat?” Since 3 goes into 300 evenly, the whole pattern must finish on the 300th slat and start all over again on the 301st. Therefore, the 301st would be red.
Exponents and Roots

Rules of Operations with Exponents

To multiply two powers with the same base, keep the base and add the exponents together.

Example:

\[ 2^2 \times 2^3 = (2 \times 2) (2 \times 2 \times 2) = 2^5 \]

or

\[ 2^2 \times 2^3 = 2^{2+3} = 2^5 \]

To divide two powers with the same base, keep the base and subtract the exponent of the denominator from the exponent of the numerator.

Example:

\[ 4^5 \div 4^2 = \frac{(4)(4)(4)(4)(4)}{(4)(4)} = 4^3 \]

or

\[ 4^5 \div 4^2 = 4^{5-2} = 4^3 \]

To raise a power to another power, multiply the exponents.

Example:

\[ (3^2)^4 = (3 \times 3)^4 \]

or

\[ (3^2)^4 = (3 \times 3)(3 \times 3)(3 \times 3)(3 \times 3) \]

or

\[ (3^2)^4 = 3^2 \times 4 = 3^8 \]
Commonly Tested Properties of Powers

Many Data Sufficiency problems test your understanding of what happens when negative numbers and fractions are raised to a power.

Raising a fraction between zero and one to a power produces a smaller result.

Example:

\[ \left( \frac{1}{2} \right)^2 = \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) = \frac{1}{4} \]

Raising a negative number to an even power produces a positive result.

Example:

\[ (-2)^2 = 4 \]

Raising a negative number to an odd power gives a negative result.

Example:

\[ (-2)^3 = -8 \]

Powers of 10

When 10 is raised to an exponent that is a positive integer, that exponent tells how many zeros the number would contain if it were written out.

Example:

Write \(10^6\) in ordinary notation.

The exponent 6 indicates that you will need six zeros after the 1: 1,000,000. That’s because \(10^6\) means six factors of 10, that is, \((10)(10)(10)(10)(10)(10)\).
To multiply a number by a power of 10, move the decimal point the same number of places to the right as the exponent (or as the number of zeros in that power of 10).

**Example:**

Multiply 0.029 by $10^3$.

The exponent is 3, so move the decimal point three places to the right.

$$(0.029)10^3 = 0029. = 29$$

If you had been told to multiply 0.029 by 1,000, you could have counted the number of zeros in 1,000 and done exactly the same thing.

Sometimes you'll have to add zeros as placeholders.

**Example:**

Multiply 0.029 by $10^6$.

Add zeros until you can move the decimal point six places to the right:

$$0.029 \times 10^6 = 0029000. = 29,000$$

To divide by a power of 10, move the decimal point the corresponding number of places to the left, inserting zeros as placeholders if necessary.

**Example:**

Divide 416.03 by 10,000

There are four zeros in 10,000, but only three places to the left of the decimal point. You'll have to insert another zero:

$$416.03 \div 10,000 = .041603 = 0.041603$$
By convention, one zero is usually written to the left of the decimal point on the GMAT. It’s a placeholder, and doesn’t change the value of the number.

**Scientific Notation**

Very large numbers (and very small decimals) take up a lot of space and are difficult to work with. So in some scientific texts, they are expressed in a shorter, more convenient form, called scientific notation.

For example, 123,000,000,000 would be written in scientific notation as $$(1.23)(10^{11})$$. 0.000000003 would be written as $$(3.0)(10^{-9})$$. (If you’re already familiar with the concept of negative exponents, you’ll know that multiplying by $$10^{-9}$$ is equivalent to dividing by $$10^9$$.)

To express a number in scientific notation, rewrite it as a product of two factors. The first factor must be greater than or equal to 1, but less than 10. The second factor must be a power of 10.

To translate a number from scientific notation to ordinary notation, use the rules for multiplying and dividing by powers of 10.

**Example:**

$$5.6 \times 10^6 = 5,600,000$$, or 5.6 million

**Rules of Operations with Roots and Radicals**

A square root of any nonnegative number $x$ is a number that, when multiplied by itself, yields $x$. Every positive number has two square roots, one positive and one negative. For instance, the positive square root of 25 is 5, because $5^2 = 25$. The negative square root of 25 is $-5$, because $(-5)^2$ also equals 25.
By convention, the radical symbol $\sqrt{}$ stands for the positive square root only. Therefore, $\sqrt{9} = 3$ only, even though both $3^2$ and $(-3)^2$ equal 9. This has important implications in Data Sufficiency.

**Example:**

What is the value of $x$?

1. $x = \sqrt{16}$
2. $x^2 = 16$

The first statement is sufficient, since there is only one possible value for $\sqrt{16}$, positive 4. The second statement is insufficient since $x$ could be 4 or $-4$.

When applying the four basic arithmetic operations, radicals (roots written with the radical symbol) are treated in much the same way as variables.

**Addition and Subtraction of Radicals**

Only like radicals can be added to or subtracted from one another.

**Example:**

$$2\sqrt{3} + 4\sqrt{2} - \sqrt{2} - 3\sqrt{3} =$$

$$(4\sqrt{2} - \sqrt{2}) + (2\sqrt{3} - 3\sqrt{3}) =$$

$$3\sqrt{2} + (-\sqrt{3}) =$$

$$3\sqrt{2} - \sqrt{3}$$

This expression cannot be simplified any further.
Chapter 2

Multiplication and Division of Radicals

To multiply or divide one radical by another, multiply or divide the numbers outside the radical signs, then the numbers inside the radical signs.

Example:

\[(6\sqrt{3})(2\sqrt{5}) = (6)(2)(\sqrt{3})(\sqrt{5}) = 12\sqrt{15}\]

Example:

\[12\sqrt{15} \div 2\sqrt{5} = \left(\frac{12}{2}\right)\left(\frac{\sqrt{15}}{\sqrt{5}}\right) = 6\sqrt{\frac{15}{5}} = 6\sqrt{3}\]

Simplifying Radicals

If the number inside the radical is a multiple of a perfect square, the expression can be simplified by factoring out the perfect square.

Example:

\[\sqrt{72} = (\sqrt{36})(\sqrt{2}) = 6\sqrt{2}\]
Fractions

The simplest way to understand the meaning of a fraction is to picture the denominator as the number of equal parts into which a whole unit is divided. The numerator represents a certain number of those equal parts.

On the left, the shaded portion is one of two equal parts that make up the whole. On the right, the shaded portion is three of four equal parts that make up the whole.

The fraction bar is interchangeable with a division sign. You can divide the numerator of a fraction by the denominator to get an equivalent decimal. However, the numerator and denominator must each be treated as a single quantity.

Example:
Evaluate \( \frac{5 + 2}{7 - 3} \)

You can’t just rewrite the fraction as \( 5 + 2 ÷ 7 - 3 \), because the numerator and the denominator are each considered distinct quantities. Instead, you would rewrite the fraction as \( (5 + 2) ÷ (7 - 3) \). The order of operations (remember PEMDAS?) tells us that operations in parentheses must be performed first.

That gives you \( 7 ÷ 4 \). Your final answer would be \( \frac{7}{4}, 1\frac{3}{4}, \) or 1.75, depending on the form of the answer choices.
Equivalent Fractions

Since multiplying or dividing a number by 1 does not change the number, multiplying the numerator and denominator of a fraction by the same nonzero number doesn’t change the value of the fraction—it’s the same as multiplying the entire fraction by 1.

Example:

Change $\frac{1}{2}$ into an equivalent fraction with a denominator of 4.

To change the denominator from 2 to 4, you’ll have to multiply it by 2. But to keep the value of the fraction the same, you’ll also have to multiply the numerator by 2.

$$\frac{1}{2} = \frac{1 \times 2}{2 \times 2} = \frac{2}{4}$$

Similarly, dividing the numerator and denominator by the same nonzero number leaves the value of the fraction unchanged.

Example:

Change $\frac{16}{20}$ into an equivalent fraction with a denominator of 10.

To change the denominator from 20 to 10, you’ll have to divide it by 2. But to keep the value of the fraction the same, you’ll have to divide the numerator by the same number.

$$\frac{16}{20} = \frac{16 \div 2}{20 \div 2} = \frac{8}{10}$$
Reducing (Canceling)

Most fractions on the GMAT are in lowest terms. That means that the numerator and denominator have no common factor greater than 1.

For example, the final answer of $\frac{8}{10}$ that we obtained in the previous example was not in lowest terms, because both 8 and 10 are divisible by 2. In contrast, the fraction $\frac{7}{10}$ is in lowest terms, because there is no factor greater than 1 that 7 and 10 have in common. To convert a fraction to its lowest terms, we use a method called reducing, or canceling. To reduce, simply divide any common factors out of both the numerator and the denominator.

**Example:**
Reduce $\frac{15}{35}$ to lowest terms.

$$\frac{15}{35} = \frac{15 \div 5}{35 \div 5} = \frac{3}{7} \text{ (because a 5 cancels out, top and bottom)}$$

The fastest way to reduce a fraction that has very large numbers in both the numerator and denominator is to find the greatest common factor and divide it out of both the top and the bottom.

**Example:**
Reduce $\frac{1040}{1080}$ to lowest terms.

$$\frac{1040}{1080} = \frac{104}{108} = \frac{52}{54} = \frac{26}{27}$$
Adding and Subtracting Fractions

You cannot add or subtract fractions unless they have the same denominator. If they don’t, you’ll have to convert each fraction to an equivalent fraction with the least common denominator. Then add or subtract the numerators (not the denominators!) and, if necessary, reduce the resulting fraction to its lowest terms.

Given two fractions with different denominators, the least common denominator is the least common multiple of the two denominators, that is, the smallest number that is evenly divisible by both denominators.

Example:

What is the least common denominator of \( \frac{2}{15} \) and \( \frac{3}{10} \)?

The least common denominator of the two fractions will be the least common multiple of 15 and 10.

\( 15 = (5)(3) \) and \( 10 = (5)(2) \), so the least common multiple of the two numbers is \( (5)(3)(2) \), or 30. That makes 30 the least common denominator of \( \frac{2}{15} \) and \( \frac{3}{10} \).

Example:

\( \frac{2}{15} + \frac{3}{10} = ? \)

As we saw in the previous example, the least common denominator of the two fractions is 30. Change each fraction to an equivalent fraction with a denominator of 30.

\[
\begin{align*}
\frac{2}{15} & = \frac{4}{30} \\
\frac{3}{10} & = \frac{9}{30}
\end{align*}
\]
Then add:
\[
\frac{4}{30} + \frac{9}{30} = \frac{13}{30}
\]

Since 13 and 30 have no common factor greater than 1, \(\frac{13}{30}\) is in lowest terms. You can’t reduce it further.

### Multiplying Fractions

To multiply fractions, multiply the numerators and multiply the denominators.

\[
\frac{5}{7} \cdot \frac{3}{4} = \frac{15}{28}
\]

Multiplying numerator by numerator and denominator by denominator is simple. But it’s easy to make careless errors if you have to multiply a string of fractions or work with large numbers. You can minimize those errors by reducing before you multiply.

**Example:**

Multiply \(\frac{10}{9} \cdot \frac{3}{4} \cdot \frac{8}{15}\)

First, cancel a 5 out of the 10 and the 15, a 3 out of the 3 and the 9, and a 4 out of the 8 and the 4:

\[
\frac{2}{9} \cdot \frac{1}{1} \cdot \frac{2}{3} = \frac{4}{9}
\]
Reciprocals

To get the reciprocal of a common fraction, turn the fraction upside-down so that the numerator becomes the denominator, and vice versa. If a fraction has a numerator of 1, the fraction’s reciprocal will be equivalent to an integer.

Example:

What is the reciprocal of $\frac{1}{25}$?

Inverting the fraction gives you the reciprocal, $\frac{25}{1}$. But dividing a number by 1 doesn’t change the value of the number.

Since $\frac{25}{1} = 25$, the reciprocal of $\frac{1}{25}$ equals 25.

Dividing Common Fractions

To divide fractions, multiply the reciprocal of the number or fraction that follows the division sign.

$$\frac{1}{2} \div \frac{3}{5} = \frac{1}{2} \left( \frac{5}{3} \right) = \frac{5}{6}$$

(The operation of division produces the same result as the inverse of multiplication.)

Example:

$$\frac{4}{3} \div \frac{4}{9} = \frac{4/9}{3/4} = \frac{36}{12} = 3$$
Comparing Positive Fractions

Given two positive fractions with the same denominator, the fraction with the larger numerator will have the larger value.

Example:
Which is greater, $\frac{3}{8}$ or $\frac{5}{8}$?

But if you’re given two positive fractions with the same numerator but different denominators, the fraction with the smaller denominator will have the larger value.

Example:
Which is greater, $\frac{3}{4}$ or $\frac{3}{8}$?

The diagrams below show two wholes of equal size. The one on the left is divided into 4 equal parts, 3 of which are shaded. The one on the right is divided into 8 equal parts, 3 of which are shaded.

$\frac{3}{8}$ is clearly greater than $\frac{3}{8}$
If neither the numerators nor the denominators are the same, you have three options. You can turn both fractions into their decimal equivalents. Or you can express both fractions in terms of some common denominator, and then see which new equivalent fraction has the largest numerator. Or you can cross-multiply the numerator of each fraction by the denominator of the other. The greater result will wind up next to the greater fraction.

Example:
Which is greater, $\frac{5}{6}$ or $\frac{7}{9}$?

Since $45 > 42$, $\frac{5}{6} > \frac{7}{9}$

Mixed Numbers and Improper Fractions

A mixed number consists of an integer and a fraction.

An improper fraction is a fraction whose numerator is greater than its denominator. To convert an improper fraction to a mixed number, divide the numerator by the denominator. The number of “whole” times that the denominator goes into the numerator will be the integer portion of the improper fraction; the remainder will be the numerator of the fractional portion.

Example:
Convert $\frac{23}{4}$ to a mixed number.

Dividing 23 by 4 gives you 5 with a remainder of 3, so $\frac{23}{4} = 5\frac{3}{4}$. 
To change a mixed number to a fraction, multiply the integer portion of the mixed number by the denominator, and add the numerator. This new number is your numerator. The denominator will not change.

Example:
Convert $2 \frac{3}{7}$ to a fraction.

$$2 \frac{3}{7} = \frac{7(2) + 3}{7} = \frac{17}{7}$$

Properties of Fractions Between $-1$ and $+1$

The reciprocal of a fraction between 0 and 1 is greater than both the original fraction and 1.

Example:
The reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$, which is greater than both 1 and $\frac{2}{3}$.

The reciprocal of a fraction between $-1$ and 0 is less than both the original fraction and $-1$.

Example:
The reciprocal of $-\frac{2}{3}$ is $-\frac{3}{2}$, or $-1 \frac{1}{2}$, which is less than both $-1$ and $-\frac{2}{3}$.
The square of a fraction between 0 and 1 is less than the original fraction.

Example:

\[
\left( \frac{1}{2} \right)^2 = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}
\]

But the square of any fraction between 0 and \(-1\) is greater than the original fraction, because multiplying two negative numbers gives you a positive product, and any positive number is greater than any negative number.

Example:

\[
\left( -\frac{1}{2} \right)^2 = \left( -\frac{1}{2} \right) \cdot \left( -\frac{1}{2} \right) = \frac{1}{4}
\]

Multiplying any positive number by a fraction between 0 and 1 gives a product smaller than the original number.

Example:

\[
6 \cdot \frac{1}{4} = \frac{6}{4} = \frac{3}{2}
\]

Multiplying any negative number by a fraction between 0 and 1 gives a product greater than the original number.

Example:

\[
(-3) \cdot \frac{1}{2} = -\frac{3}{2}
\]
Decimals

Converting Decimals

It’s easy to convert decimals to common fractions, and vice versa. Any decimal fraction is equivalent to some common fraction with a power of 10 in the denominator.

To convert a decimal between 0 and 1 to a fraction, determine the place value of the last nonzero digit, and set this as the denominator. Then use all the digits of the decimal number as the numerator, ignoring the decimal point. Finally, if necessary, reduce the fraction to its lowest terms.

Example:

Convert 0.875 to a fraction in lowest terms.

The last nonzero digit is the 5, which is in the thousandths’ place. So the denominator of the common fraction will be 1,000.

The numerator will be 875: \( \frac{875}{1000} \)

(You can ignore the zero to the left of the decimal point, since there are no nonzero digits to its left; it’s just a “placeholder.”)

Both 875 and 1,000 contain a factor of 25. Canceling it out leaves you with \( \frac{35}{40} \). Reducing that further by a factor of 5 gives you \( \frac{7}{8} \), which is in lowest terms.
To convert a fraction to a decimal, simply divide the numerator by the denominator.

Example:

What is the decimal equivalent of \( \frac{4}{5} \)?

\[
4 \div 5 = 0.8
\]

Comparing Decimals

Knowing place values allows you to assess the relative values of decimals.

Example:

Which is greater, 0.254 or 0.3?

Of course, 254 is greater than 3. But 0.3 = \( \frac{3}{10} \), which is equivalent to \( \frac{300}{1000} \), while 0.254 is equivalent to only \( \frac{254}{1000} \). Since \( \frac{300}{1000} > \frac{254}{1000} \), 0.3 is greater than 0.254.

Here’s the simplest way to compare decimals: Add zeros after the last digit to the right of the decimal point in each decimal fraction until all the decimals you’re comparing have the same number of digits. Essentially, what you’re doing is giving all the fractions the same denominator so that you can just compare their numerators.
Example:

Arrange in order from smallest to largest: 0.7, 0.77, 0.07, 0.707, and 0.077.

0.707 and 0.077 end at the third place to the right of the decimal point—the thousandths place. Add zeros after the last digit to the right of the decimal point in each of the other fractions until you reach the thousandths’ place:

\[
\begin{align*}
0.7 &= 0.700 &= \frac{700}{1000} \\
0.77 &= 0.770 &= \frac{770}{1000} \\
0.07 &= 0.070 &= \frac{70}{1000} \\
0.707 &= \frac{707}{1000} \\
0.077 &= \frac{77}{1000}
\end{align*}
\]

\[
\frac{700}{1000} < \frac{77}{1000} < \frac{700}{1000} < \frac{707}{1000} < \frac{770}{1000}
\]

Therefore, \(0.07 < 0.077 < 0.7 < 0.707 < 0.77\).
Estimation and Rounding on the GMAT

You should be familiar and comfortable with the practice of “rounding off” numbers. To round off a number to a particular place, look at the digit immediately to the right of that place. If the digit is 0, 1, 2, 3, or 4, don’t change the digit that is in the place to which you are rounding. If it is 5, 6, 7, 8, or 9, change the digit in the place to which you are rounding to the next higher digit. Replace all digits to the right of the place to which you are rounding with zeros.

For example, to round off 235 to the tens’ place, look at the units’ place. Since it is occupied by a 5, you’ll round the 3 in the tens’ place up to a 4, giving you 240. If you had been rounding off 234, you would have rounded down to the existing 3 in the tens’ place; that would have given you 230.

Example:

Round off 675,978 to the hundreds’ place.

The 7 in the tens’ place means that you will have to round the hundreds’ place up. Since there is a 9 in the hundreds’ place, you’ll have to change the thousands’ place as well. Rounding 675,978 to the hundreds’ place gives you 676,000.

Rounding off large numbers before calculation will allow you quickly to estimate the correct answer.

Estimating can save you valuable time on many GMAT problems. But before you estimate, check the answer choices to see how close they are. If they are relatively close together, you’ll have to be more accurate than if they are farther apart.
Percents

The word percent means “hundredths,” and the percent sign, %, means $\frac{1}{100}$. For example, 25% means $25 \left( \frac{1}{100} \right) = \frac{25}{100}$. (Like the division sign, the percent sign evolved from the fractional relationship; the slanted bar in a percent sign represents a fraction bar.)

Percents measure a part-to-whole relationship with an assumed whole equal to 100. The percent relationship can be expressed as $\frac{\text{Part}}{\text{Whole}}$ (100%). For example, if $\frac{1}{4}$ of a rectangle is shaded, the percent of the rectangle that is shaded is $\frac{1}{4}$ (100%) = 25%.

Like fractions, percents express the relationship between a specified part and a whole. In fact, by plugging the part and whole from the shaded rectangle problem into the fraction and decimal versions of the part-whole equation, you can verify that 25%, $\frac{25}{100}$, and 0.25 are simply different names for the same part-whole relationship.

Translating English to Math in Part-Whole Problems

On the GMAT, many fractions and percents appear in word problems. You’ll solve the problems by plugging the numbers you’re given into some variation of one of the three basic formulas:

$$\frac{\text{Part}}{\text{Whole}} = \text{Fraction}$$

$$\frac{\text{Part}}{\text{Whole}} = \text{Decimal}$$

$$\frac{\text{Part}}{\text{Whole}} \times (100) = \text{Percent}$$
To avoid careless errors, look for the key words *is* and *of*. *Is* (or *are*) often introduces the part, while *of* almost invariably introduces the whole.

**Properties of 100%**

Since the percent sign means \( \frac{1}{100} \), 100% means \( \frac{100}{100} \), or one whole. The key to solving some GMAT percent problems is to recognize that all the parts add up to one whole: 100%.

**Example:**

All 1,000 registered voters in Smithtown are Democrats, Republicans, or Independents. If 75% of the registered voters are Democrats, and 5% are Independents, how many are Republicans?

75% + 5%, or 80% of the 1,000 registered voters are either Democrats or Independents. The three party affiliations together must account for 100% of the voters; thus, the percentage of Republicans must be 100% – 80%, or 20%. Thus, the number of Republicans must be 20% of 1,000, which is 20% (1,000), or 200.

Multiplying or dividing a number by 100% is just like multiplying or dividing by 1; it doesn’t change the value of the original number.
Converting Percents

To change a fraction to its percent equivalent, multiply by 100%.

Example:
What is the percent equivalent of \( \frac{5}{8} \)?

\[
\frac{5}{8} \times 100\% = \frac{500\%}{8} = 62\frac{1}{2}\%
\]

To change a decimal fraction to a percent, you can use the rules for multiplying by powers of 10. Move the decimal point two places to the right and insert a percent sign.

Example:
What is the percent equivalent of 0.17?

\[
0.17 = 0.17 \times 100\% = 17\%
\]

To change a percent to its fractional equivalent, divide by 100%.

Example:
What is the common fraction equivalent of 32%?

\[
32\% = \frac{32\%}{100\%} = \frac{8}{25}
\]

To convert a percent to its decimal equivalent, use the rules for dividing by powers of 10—just move the decimal point two places to the left.

Example:
What is the decimal equivalent of 32%?

\[
32\% = \frac{32\%}{100\%} = \frac{32}{100} = 0.32
\]
When you divide a percent by another percent, the percent sign “drops out,” just as you would cancel out a common factor.

Example:

\[
\frac{100\%}{5\%} = \frac{100}{5} = 20
\]

Translation: There are 20 groups of 5% in 100%.

But when you divide a percent by a regular number (not by another percent), the percent sign remains.

Example:

\[
\frac{100\%}{5} = 20\%
\]

Translation: One-fifth of 100% is 20%.
Common Percent Equivalents

As you can see, changing percents to fractions, or vice versa, is pretty straightforward. But it does take a second or two that you might spend more profitably doing other computations or setting up another GMAT math problem. Familiarity with the following common equivalents will save you time.

<table>
<thead>
<tr>
<th>Fraction</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{20}$</td>
<td>5%</td>
</tr>
<tr>
<td>$\frac{1}{12}$</td>
<td>$8\frac{1}{3}$%</td>
</tr>
<tr>
<td>$\frac{1}{10}$</td>
<td>10%</td>
</tr>
<tr>
<td>$\frac{1}{8}$</td>
<td>$12\frac{1}{2}$%</td>
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<td>$\frac{1}{6}$</td>
<td>$16\frac{2}{3}$%</td>
</tr>
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<td>20%</td>
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<tr>
<td>$\frac{1}{4}$</td>
<td>25%</td>
</tr>
<tr>
<td>$\frac{3}{10}$</td>
<td>30%</td>
</tr>
<tr>
<td>$\frac{1}{3}$</td>
<td>$33\frac{1}{3}$%</td>
</tr>
<tr>
<td>$\frac{3}{8}$</td>
<td>$37\frac{1}{2}$%</td>
</tr>
<tr>
<td>$\frac{2}{5}$</td>
<td>40%</td>
</tr>
</tbody>
</table>
Using the Percent Formula to Solve Percent Problems

You can solve most percent problems by plugging the given data into the percent formula:

\[
\frac{\text{part}}{\text{Whole}} (100\%) = \text{Percent}
\]

Most percent problems give you two of the three variables and ask for the third.

Example:

Ben spends $30 of his annual gardening budget on seed. If his total annual gardening budget is $150, what percentage of his budget does he spend on seed?

This problem specifies the whole ($150) and the part ($30) and asks for the percentage. Plugging those numbers into the percent formula gives you:

\[
\text{Percent} = \frac{30}{150} (100\%) = \frac{1}{5} (100\%) = 20
\]

Ben spends 20% of his annual gardening budget on seed.

Percent Increase and Decrease

When the GMAT tests percent increase or decrease, use the formulas:

\[
\text{% increase} = \frac{\text{increase (100\%)}}{\text{original}} \quad \text{or} \quad \text{% decrease} = \frac{\text{decrease (100\%)}}{\text{original}}
\]

To find the difference, just subtract the original from the new. Note that the "original" is the base from which change occurs. It may or may not be the first number mentioned in the problem.
Example:

Two years ago, 450 seniors graduated from Inman High School. Last year, 600 seniors graduated. By what percentage did the number of graduating seniors increase?

The original is the figure from the earlier time (two years ago): 450. The increase is 600 – 450, or 150. So the percentage increase is \( \frac{150}{450} \times 100\% = 33\frac{1}{3}\% \).

Example:

If the price of a $120 dress is increased by 25%, what is the new selling price?

To find the new whole, you'll first have to find the amount of increase. The original whole is $120, and the percent increase is 25%. Plugging in, we find that

\[
\frac{\text{increase}}{120} \times 100\% = 25\%
\]

\[
\frac{\text{increase}}{120} = \frac{25}{100}
\]

\[
\frac{\text{increase}}{120} = \frac{1}{4}
\]

\[
\text{increase} = \frac{120}{4}
\]

\[
\text{increase} = 30
\]

The amount of increase is $30, so the new selling price is $120 + $30, or $150.
Multistep Percent Problems

On some difficult problems, you'll be asked to find more than one percent, or to find a percent of a percent. Be careful: You can't add percents of different wholes.

Example:

The price of an antique is reduced by 20 percent, and then this price is reduced by 10 percent. If the antique originally cost $200, what is its final price?

The most common mistake in this kind of problem is to reduce the original price by a total of 20% + 10%, or 30%. That would make the final price 70 percent of the original, or 70% ($200) = $140. This is not the correct answer. In this example, the second (10%) price reduction is taken off of the first sale price—the new whole, not the original whole.

To get the correct answer, first find the new whole. You can find it by calculating either $200 - (20\% \text{ of } $200)$ or 80\% ($200). Either way, you will find that the first sale price is $160. That price then has to be reduced by 10\%. Either calculate $160 - (10\% \text{ of } $160)$ or 90\%($160). In either case, the final price of the antique is $144.

Picking Numbers with Percents

Certain types of percent problems lend themselves readily to the alternative technique of Picking Numbers. These include problems in which no actual values are mentioned, just percents. If you assign values to the percents you are working with, you'll find the problem less abstract.

You should almost always pick 100 in percent problems, because it's relatively easy to find percentages of 100.
Example:

The price of a share of company A’s stock fell by 20 percent two weeks ago and by another 25 percent last week to its current price. By what percent of the current price does the share price need to rise in order to return to its original price?

(A) 45%
(B) 55%
(C) 66 \(\frac{2}{3}\) %
(D) 75%
(E) 82%

Pick a value for the original price of the stock. Since it is a percent question, picking $100 will make the math easy. The first change in the price of the stock was by 20% of $100, or $20, making the new price $100 - $20 = $80. The price then fell by another 25%.

25% is the same as \(\frac{1}{4}\), and \(\frac{1}{4}\) of $80 is $20. Therefore, the current price is $80 - $20 = $60. To return to its original price, the stock needs to rise from $60 to $100, that is, by $100 - $60 = $40. $40 is what percent of the current price, $60?

\[
\frac{40}{60} (100\%) = \frac{2}{3} (100\%) = 66 \frac{2}{3} \%
\]
Percent Word Problems

Percent problems are often presented as word problems. We have already seen how to identify the percent, the part, and the whole in simple percent word problems. Here are some other terms that you are likely to encounter in more complicated percent word problems:

**Profit** made on an item is the seller’s price minus the costs to the seller. If a seller buys an item for $10 and sells it for $12, he or she has made $2 profit. The percent of the selling price that is profit is:

\[
\text{Profit} \quad \frac{\text{Original Selling Price}}{(100\%)} = \frac{\$2}{\$12} (100\%) = 16\frac{2}{3}\%.
\]

A **discount** on an item is the original price minus the reduced price. If an item that usually sells for $20 is sold for $15, the discount is $5. Discount is often represented as a percentage of the original price. In this case, the percentage discount is:

\[
\text{Discount} \quad \frac{\text{Original Price}}{(100\%)} = \frac{\$5}{\$20} = 25\%.
\]

The **sale price** is the final price after discount or decrease.

Occasionally, percent problems will involve **interest**. Interest is given as a percent per unit time, such as 5% per month. The sum of money invested is the **principal**. The most common type of interest you will see is **simple interest**. In simple interest, the interest payments received are kept separate from the principal.

**Example:**

If an investor invests $100 at 20 percent simple annual interest, how much does he or she have at the end of 3 years?

The principal of $100 yields 20% interest every year. 20% of $100 is $20, so after three years the investor will have 3 years of interest, or $60, plus the principal, for a total of $160.
In compound interest, the money earned as interest is reinvested. The principal grows after every interest payment received.

Example:

If an investor invests $100 at 20% compounded annually, how much does he or she have at the end of 3 years?

The first year the investor earns 20% of $100 = $20. So, after one year he or she has $100 + $20 = $120.

The second year the investor earns 20% of $120 = $24. So, after two years he or she has $120 + $24 = $144.

The third year the investor earns 20% of $144 = $28.80. So, after 3 years he or she has $144 + $28.80 = $172.80.

Percents and Data Sufficiency

Data Sufficiency questions (covered in Session 2) test your knowledge of percents in a different way. The crux of these problems, as a rule, is finding all the pieces of the percent formula. You can use the percent formula to pinpoint exactly what you need to achieve sufficiency.

Example:

By what percent did the price of stock $X$ increase?

(1) The price after the increase was $12.

(2) The stock increased in price by $1.50.

To prove sufficiency you would have to be capable of filling in all parts of the equation. Statement (1) informs you of the price after the increase. This does not give you either the amount of increase or the original price, so it is not sufficient. Statement (2) informs you of the increase in price, but not the original price, so it, too, is not sufficient. Combining the statements, however, gives you the increase in price, $1.50, and the original price, $12.00 – $1.50 = $10.50. So the correct answer is choice (C).
Ratios

A ratio is the proportional relationship between two quantities. The ratio, or relationship, between two numbers, for example, 10 and 15, may be expressed with a colon between the two numbers (10:15), in words (“the ratio of 10 to 15”), or as a common fraction \( \frac{10}{15} \).

To translate a ratio in words to numbers separated by a colon, replace to with a colon.

To translate a ratio in words to a fractional ratio, use whatever follows the word of as the numerator and whatever follows the word to as the denominator. For example, if we had to express the ratio of glazed doughnuts to chocolate doughnuts in a box of doughnuts that contained 5 glazed and 7 chocolate doughnuts, we would do so as \( \frac{5}{7} \).

Note that the fraction \( \frac{5}{7} \) does not mean that \( \frac{5}{7} \) of all the doughnuts are glazed doughnuts. There are \( 5 + 7 = 12 \) doughnuts all together, so of the doughnuts, \( \frac{5}{12} \) are glazed. The \( \frac{5}{7} \) ratio merely indicates the proportion of glazed to chocolate doughnuts. For every five glazed doughnuts, there are seven chocolate doughnuts.

Treating ratios as fractions usually makes computation easier. Like fractions, ratios often require division. And, like fractions, ratios can be reduced to lowest terms.

Example:

Joe is 16 years old, and Mary is 12 years old. Express the ratio of Joe's age to Mary's age in lowest terms.

The ratio of Joe's age to Mary's age is \( \frac{16}{12} = \frac{4}{3} \), or 4:3.
CHAPTER 3

Part:Whole Ratios

In a part:whole ratio, the “whole” is the entire set (for instance, all the workers in a factory), while the “part” is a certain subset of the whole (for instance, all the female workers in the factory).

In GMAT ratio question stems, the word fraction generally indicates a part:whole ratio. “What fraction of the workers are female?” means “What is the ratio of the number of female workers to the total number of workers?”

Example:

The sophomore class at Milford Academy consists of 15 boys and 20 girls. What fraction of the sophomore class is female?

The following three statements are equivalent:

1. $\frac{4}{7}$ of the sophomores are female.
2. 4 out of every 7 sophomores are female.
3. The ratio of female sophomores to total sophomores is 4:7.

Ratio vs. Actual Number

Ratios are usually reduced to their simplest form (that is, to lowest terms). If the ratio of men to women in a room is 5:3, you cannot necessarily infer that there are exactly five men and three women.

If you knew the total number of people in the room, in addition to the male to female ratio, you could determine the number of men and the number of women in the room. For example, suppose you know that there are 32 people in the room. If the male to female ratio is 5 to 3, then the ratio of males to the total is 5:(5 + 3), which is 5:8. You can set
up an equation as $\frac{5}{8} = \frac{\text{# of males in room}}{32}$. Solving, you will find that the number of males in the room is 20.

**Example:**

The ratio of domestic sales revenues to foreign sales revenues of a certain product is 3:5. What fraction of the total sales revenues comes from domestic sales?

At first, this question may look more complicated than the previous example. You have to convert from a part:part ratio to a part:whole ratio (the ratio of domestic sales revenues to total sales revenues). And you’re not given actual dollar figures for domestic or foreign sales. But since all sales are either foreign or domestic, “total sales revenues” must be the sum of the revenues from domestic and foreign sales. You can convert the given ratio to a part:whole ratio, because the sum of the parts equals the whole.

Although it’s impossible to determine dollar amounts for the domestic, foreign, or total sales revenues from the given information, the 3:5 ratio tells you that of every $8 in sales revenues, $3 come from domestic sales and $5 from foreign sales. Therefore, the ratio of domestic sales revenues to total sales revenues is 3:8, or $\frac{3}{8}$.

You can convert a part:part ratio to a part:whole ratio (or vice versa) only if there are no missing parts and no overlap among the parts; that is, if the whole is equal to the sum of the parts.
This concept is often tested in Data Sufficiency:

**Example:**

In a certain bag, what is the ratio of the number of red marbles to the total number of marbles?

1. The ratio of the number of red marbles to the number of blue marbles in the bag is 3:5.
2. There are only red and blue marbles in the bag.

In this case, statement (1), by itself, is insufficient. You cannot convert a part to part ratio (red marbles to blue marbles) to a part to whole ratio (red marbles to all marbles) because you don't know whether there were any other colored marbles in the bag. Only when you combine the two statements do you have enough information to answer the question, so the answer is (C).

**Example:**

Of the 25 people in Fran’s apartment building, what is the ratio of people who use the roof to total residents?

1. There are 9 residents who use the roof for tanning and 8 residents who use the roof for gardening.
2. The roof is only used by tanners and gardeners.

In this question, we do not know if there is any overlap between tanners and gardeners. How many, if any, residents do both? Since we don’t know, the answer is (E).
Ratios of More Than Two Terms

Most of the ratios that you’ll see on the GMAT have two terms. But it is possible to set up ratios with more than two terms. These ratios express more relationships, and therefore convey more information, than two-term ratios. However, most of the principles discussed so far with respect to two-term ratios are just as applicable to ratios of more than two terms.

Example:

The ratio of $x$ to $y$ is $5:4$. The ratio of $y$ to $z$ is $1:2$. What is the ratio of $x$ to $z$?

We want the $y$’s in the two ratios to equal each other, because then we can combine the $x:y$ ratio and the $y:z$ ratio to form the $x:y:z$ ratio that we need to answer this question. To make the $y$’s equal, we can multiply the second ratio by 4. When we do so, we must perform the multiplication on both components of the ratio. Since a ratio is a constant proportion, it can be multiplied or divided by any number without losing its meaning, as long as the multiplication and division are applied to all the components of the ratio. In this case, we find that the new ratio for $y$ to $z$ is $4:8$. We can combine this with the first ratio to find a new $x$ to $y$ to $z$ ratio of $5:4:8$. Therefore, the ratio of $x$ to $z$ is $5:8$. 
Chapter 3

Rates

A rate is a special type of ratio. Instead of relating a part to the whole, or to another part, a rate relates one kind of quantity to a completely different kind. When we talk about rates, we usually use the word *per*, as in “miles per hour,” “cost per item,” etc. Since *per* means “for one” or “for each,” we express the rates as ratios reduced to a denominator of 1.

Speed

The most commonly tested rate on the GMAT is speed. This is usually expressed in miles or kilometers per hour. The relationship between speed, distance, and time is given by the formula \( \text{Speed} = \frac{\text{Distance}}{\text{Time}} \) which can be rewritten two ways: \( \text{Time} = \frac{\text{Distance}}{\text{Speed}} \) and \( \text{Distance} = (\text{Speed})(\text{Time}) \).

Any time you can find two out of the three elements in this equation, you can find the third.

For example, if a car travels 300 miles in 5 hours, it has averaged \( \frac{300 \text{ miles}}{5 \text{ hours}} = 60 \text{ miles per hour} \). (Note that speeds are usually expressed as averages because they are not necessarily constant. For instance, in the previous example, the car traveled 300 miles in 5 hours. It moved at an “average speed” of 60 miles per hour, but probably not at a constant speed of 60 miles per hour.)

Likewise, a rearranged version of the formula can be used to solve for missing speed or time.
Example:
How far do you drive if you travel for 5 hours at 60 miles per hour?

Distance = (Speed)(Time)
Distance = (60 mph)(5 hours)
Distance = 300 miles

Example:
How much time does it take to drive 300 miles at 60 miles per hour?

Time = \frac{\text{Distance}}{\text{Speed}}
Time = \frac{300 \text{ miles}}{60 \text{ mph}}
Time = 5 \text{ hours}

Other Rates
Speed is not the only rate that appears on the GMAT. For instance, you might get a word problem involving liters per minute or cost per unit. All rate problems, however, can be solved using the speed formula and its variants by conceiving of “speed” as “rate,” and “distance” as “quantity.”

Example:
How many hours will it take to fill a 500-liter tank at a rate of 2 liters per minute?

Plug the numbers into our rate formula:

Time = \frac{\text{Quantity}}{\text{Rate}}
Time = \frac{500 \text{ Liters}}{2 \text{ Liters Per Minute}}
Time = 250 \text{ minutes}
Now convert 250 minutes to hours: 250 minutes ÷ 60 minutes per hour = $4\frac{1}{6}$ hours to fill the pool. (As you can see from this problem, GMAT Problem Solving questions test your ability to convert minutes into hours and vice versa. Pay close attention to what unit the answer choice must be.)

In some cases, you should use proportions to answer rate questions.

**Example:**

If 350 widgets cost $20, how much will 1,400 widgets cost at the same rate?

Set up a proportion:

\[
\frac{\text{Number of Widgets}}{\text{Cost}} = \frac{350 \text{ Widgets}}{\$20} = \frac{1400 \text{ Widgets}}{\$x}
\]

Solving, you will find that \(x = 80\).

So, 1,400 widgets will cost $80 at that rate.

**Combined Rate Problems**

Rates can be added.

**Example:**

Nelson can mow 200 square meters of lawn per hour. John can mow 100 square meters of lawn per hour. Working simultaneously but independently, how many hours will it take Nelson and John to mow 1,800 square meters of lawn?

Add Nelson’s rate to John’s rate to find the combined rate.
200 meters per hour + 100 meters per hour = 300 meters per hour.

Divide the total lawn area, 1,800 square meters, by the combined rate, 300 square meters per hour, to find the number of required hours, 6.

**Work Problems (Given Hours per Unit of Work)**

The work formula can be used to find out how long it takes a number of people working together to complete a task. Let’s say we have three people. The first takes \(a\) units of time to complete the job, the second \(b\) units of time to complete the job, and the third \(c\) units of time. If the time it takes all three working together to complete the job is \(T\), then

\[
\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{T}.
\]

**Example:**

John can weed the garden in 3 hours. If Mary can weed the garden in 2 hours, how long will it take them to weed the garden at this rate, working independently?

Set John’s time per unit of work as \(a\), and Mary’s time per unit of work as \(b\). (There is no need for the variable \(c\), since there are only two people.) Plugging in, you find that:

\[
\frac{1}{3} + \frac{1}{2} = \frac{1}{T}
\]

\[
\frac{2}{6} + \frac{3}{6} = \frac{1}{T}
\]

\[
\frac{5}{6} = \frac{1}{T}
\]

\[T = \frac{6}{5} \text{ hours}\]
CHAPTER 3

Work Formula

We can use the above equation, \( \frac{1}{a} + \frac{1}{b} = \frac{1}{T} \), to derive the work formula, a convenient formula to use on Test Day.

\[
\frac{1}{a} + \frac{1}{b} = \frac{1}{T}
\]

\[
(ab)\left(\frac{1}{a} + \frac{1}{b}\right) = \left(\frac{1}{T}\right)(ab)
\]

\[
\frac{ab}{a} + \frac{ab}{b} = \frac{ab}{T}
\]

\[
b + a = \frac{ab}{T}
\]

\[
T(b + a) = \left(\frac{ab}{T}\right)T
\]

\[
T(b + a) = ab
\]

\[
T = \frac{ab}{a + b}
\]

This last equation is the work formula.

Here, \( a = \) the amount of time it takes person \( a \) to complete the job and \( b = \) the amount of time it takes person \( b \) to complete the job.

Example:

Let’s use the same example from above: John takes 3 hours to weed the garden and Mary takes 2 hours to weed the same garden. How long will it take them to weed the garden together?

Work formula \( \frac{a \times b}{a + b} = \frac{3 \times 2}{3 + 2} = \frac{6}{5} \) hours
Averages

The average of a group of numbers is defined as the sum of the terms divided by the number of terms.

$$\text{Average} = \frac{\text{Sum of Terms}}{\text{Number of Terms}}$$

This equation can be rewritten two ways:

Number of Terms = \frac{\text{Sum of Terms}}{\text{Average}}

Sum of Terms = (Number of Terms)(Average)

Thus, any time you have two out of the three values (average, sum of terms, number of terms), you can find the third.

Example:

Henry buys three items costing $2.00, $1.75, and $1.05. What is the average price (arithmetic mean) of the three items? (Don’t let the phrase arithmetic mean throw you; it’s just another term for average.)

$$\text{Average} = \frac{\text{Sum of Terms}}{\text{Number of Terms}}$$

$$\text{Average} = \frac{\$2.00 + \$1.75 + \$1.05}{3}$$

$$\text{Average} = \frac{\$4.80}{3}$$

$$\text{Average} = \$1.60$$
Example:
June pays an average price of $14.50 for 6 articles of clothing. What is the total price of all 6 articles?

\[
\text{Sum of Terms} = \text{(Average)} \times \text{(Number of Items)}
\]

\[
\text{Sum of Terms} = ($14.50)(6)
\]

\[
\text{Sum of Terms} = $87.00
\]

Example:
The total weight of the licorice sticks in a jar is 30 ounces. If the average weight of each licorice stick is 2 ounces, how many licorice sticks are there in the jar?

\[
\text{Number of Items} = \frac{\text{Sum of Terms}}{\text{Average}}
\]

\[
\text{Number of Items} = \frac{30 \text{ ounces}}{2 \text{ ounces}}
\]

\[
\text{Number of Items} = 15
\]

Using the Average to Find a Missing Number
If you’re given the average, the total number of terms, and all but one of the actual numbers, you can find the missing number.

Example:
The average annual rainfall in Boynton for 1976–1979 was 26 inches per year. Boynton received 24 inches of rain in 1976, 30 inches in 1977, and 19 inches in 1978. How many inches of rainfall did Boynton receive in 1979?

You know that total rainfall equals \(24 + 30 + 19 + \text{(number of inches of rain in 1979)}\).
You know that the average rainfall was 26 inches per year.
You know that there were 4 years.
So, plug these numbers into any of the three expressions of the average formula to find that
\[
\text{Sum of Terms} = \text{(Average)} \times \text{(Number of Terms)}
\]
\[
24 + 30 + 19 + \text{Inches in 1979} = (26)(4)
\]
\[
73 + \text{Inches in 1979} = (26)(4)
\]
\[
73 + \text{Inches in 1979} = 104
\]
\[
\text{Inches in 1979} = 31
\]

Another Way to Find a Missing Number: The Concept of “Balanced Value”

Another way to find a missing number is to understand that the sum of the differences between each term and the mean of the set must equal zero. Plugging in the numbers from the previous problem, for example, we find that
\[
(24 - 26) + (30 - 26) + (19 - 26) + \text{(inches in 1979 - 26)} = 0
\]
\[
(-2) + (4) + (-7) + \text{(inches in 1979 - 26)} = 0
\]
\[
-5 + \text{(inches in 1979 - 26)} = 0
\]
\[
\text{inches in 1979} = 31
\]

It may be easier to comprehend why this is true by visualizing a balancing, or weighting, process. The combined distance of the numbers above the average from the mean must be balanced with the combined distance of the numbers below the average from the mean.
Example:
The average of 63, 64, 85, and $x$ is 80. What is the value of $x$?

Think of each value in terms of its position relative to the average, 80.

63 is 17 less than 80.
64 is 16 less than 80.
85 is 5 greater than 80.

So these three terms are a total of $17 + 16 - 5$, or 28, less than the average. Therefore, $x$ must be 28 greater than the average to restore the balance at 80. So $x = 28 + 80 = 108$.

**Average of Consecutive, Evenly Spaced Numbers**

When consecutive numbers are evenly spaced, the average is the middle value. For example, the average of consecutive integers 6, 7, and 8 is 7.

If there is an even number of evenly spaced numbers, there is no single middle value. In that case, the average is midway between (that is, the average of) the middle two values. For example, the average of 5, 10, 15, and 20 is 12.5, midway between the middle values 10 and 15.

Note that not all consecutive numbers are evenly spaced. For instance, consecutive prime numbers arranged in increasing order are not evenly spaced. But you can use the handy technique of finding the middle value whenever you have consecutive integers, consecutive odd or even numbers, consecutive multiples of an integer, or any other consecutive numbers that are evenly spaced.
Combining Averages

When there is an equal number of items in each set, and only when there is an equal number of items in each set, you can average averages.

For example, suppose there are two bowlers, and you must find their average score per game. One has an average score per game of 100, and the other has an average score per game of 200. If both bowlers bowled the same number of games, you can average their averages to find their combined average. Suppose they both bowled 4 games. Their combined average will be equally influenced by both bowlers. Hence, their combined average will be the average of 100 and 200. We can find this quickly by remembering that the quantity above the average and the quantity below the average must be equal. Therefore, the average will be halfway between 100 and 200, which is 150. Or, we could solve using our average formula:

\[
\text{Average} = \frac{\text{Sum of Terms}}{\text{Number of Items}} = \frac{4(100) + 4(200)}{8} = 150
\]

However, if the bowler with the average score of 100 had bowled 4 games, and the bowler with the 200 average had bowled 16 games, the combined average would be weighted further towards 200 than towards 100, to reflect the greater influence of the 200 bowler than the 100 bowler upon the total. This is known as a weighted average.

Again, you can solve this by using the concept of a balanced average or by using the average formula.

Since the bowler bowling an average score of 200 bowled \( \frac{4}{5} \) of the games, the combined average will be \( \frac{4}{5} \) of the distance along the number line between 100 and 200, which is 180. Or, you can plug numbers into an average formula to find that
Average = \frac{\text{Sum of Terms}}{\text{Number of Items}}

Average = \frac{4(100) + 16(200)}{20}

Average = \frac{400 + 3200}{20}

Average = 180

**Averages and Data Sufficiency**

For Data Sufficiency average questions, you will have to scan the statements for any two elements of the average formula, from which you will know that you can find the third.

**Example:**

If the receipts for a matinee performance at the Granada Theater totaled $2,400, how many tickets were sold for that performance?

(1) The average price of a ticket sold was $7.50.

(2) All tickets sold cost either $10.00 or $6.00.

Use the average formula: \( \text{Average} = \frac{\text{Sum of Terms}}{\text{Number of Items}} \). In this case, you already know the Sum of Terms (total receipts = $2,400). All that you need to find the Number of Items (number of tickets sold) is the other part of the equation: the average price of a ticket sold. Statement (1) gives you this so it is sufficient. Since you don’t know how many $10.00 vs. $6.00 tickets were sold, statement (2) does not give you the number of tickets sold, and so is not sufficient. The answer then is choice (A).
Algebraic Terms

Variable: a letter or symbol representing an unknown quantity.

Constant (term): a number not multiplied by any variable(s).

Term: a numerical constant; also, the product of a numerical constant and one or more variables.

Coefficient: the numerical constant by which one or more variables are multiplied. The coefficient of $3x^2$ is 3. A variable (or product of variables) without a numerical coefficient, such as $z$ or $xy^3$, is understood to have a coefficient of 1.

Algebraic expression: an expression containing one or more variables, one or more constants, and possibly one or more operation symbols. In the case of the expression $x$, there is an implied coefficient of 1. An expression does not contain an equal sign. $x$, $3x^2 + 2x$, and $\frac{7x + 1}{3x^2 - 14}$ are all algebraic expressions.

Monomial: an algebraic expression with only one term. To multiply monomials, multiply the coefficients and the variables separately. $2a \times 3a = (2 \times 3)(a \times a) = 6a^2$.

Polynomial: the general name for an algebraic expression with more than one term.

Algebraic equation: two algebraic expressions separated by an $=$ sign, or one algebraic expression separated from a number by an $=$ sign.
Basic Operations

Combining Like Terms

The process of simplifying an expression by adding together or subtracting terms that have the same variable factors is called combining like terms.

Example:

Simplify the expression $2x - 5y - x + 7y$.

$2x - 5y - x + 7y = (2x - x) + (7y - 5y) = x + 2y$

Notice that the commutative, associative, and distributive laws that govern arithmetic operations with ordinary numbers also apply to algebraic terms and polynomials.

Adding and Subtracting Polynomials

To add or subtract polynomials, combine like terms.

$(3x^2 + 5x + 7) - (x^2 + 12) = (3x^2 - x^2) + 5x + (7 - 12) = 2x^2 + 5x - 5$
Factoring Algebraic Expressions

Factoring a polynomial means expressing it as a product of two or more simpler expressions. Common factors can be factored out by using the distributive law.

Example:
Factor the expression $2a + 6ac$.

The greatest common factor of $2a + 6ac$ is $2a$. Using the distributive law, you can factor out $2a$, so that the expression becomes $2a(1 + 3c)$.

Example:
All three terms in the polynomial $3x^3 + 12x^2 - 6x$ contain a factor of $3x$. Pulling out the common factor yields $3x(x^2 + 4x - 2)$. 
Advanced Operations

Substitution

Substitution, a process of plugging values into equations, is used to evaluate an algebraic expression, or to express it in terms of other variables.

Replace every variable in the expression with the number or quantity you are told is its equivalent. Then carry out the designated operations, remembering to follow the order of operations (PEMDAS).

Example:

Express \( \frac{a - b^2}{b - a} \) in terms of \( x \) if \( a = 2x \) and \( b = 3 \).

Replace every \( a \) with \( 2x \) and every \( b \) with \( 3 \):

\[
\frac{a - b^2}{b - a} = \frac{2x - 9}{3 - 2x}
\]

Without more information, you can’t simplify or evaluate this expression further.

Solving Equations

When you manipulate any equation, always do the same thing to both sides of the equal sign. Otherwise, the two sides of the equation will no longer be equal.

To solve an algebraic equation without exponents for a particular variable, you have to manipulate the equation until that variable is on one side of the equal sign, with all numbers or other variables on the other side. You can perform addition, subtraction, or multiplication; you can also perform division, as long as the quantity by which you are dividing does not equal zero.
Typically, at each step of the process, you'll try to isolate the variable by using the reverse of whatever operation has been applied to the variable. For example, in solving the equation \( n + 6 = 10 \) for \( n \), you have to get rid of the 6 that has been added to the \( n \). You do that by subtracting 6 from both sides of the equation: \( n + 6 - 6 = 10 - 6 \), so \( n = 4 \).

**Example:**

If \( 4x - 7 = 2x + 5 \), what is the value of \( x \)?

Start by adding 7 to both sides. This gives us \( 4x = 2x + 12 \). Now subtract \( 2x \) from both sides. This gives us \( 2x = 12 \). Finally, let's divide both sides by 2. This gives us \( x = 6 \).

**Inequalities**

There are two differences between solving an inequality (such as \( 2x < 5 \)) and solving an equation (such as \( 2x - 5 = 0 \)).

First, the solution to an inequality is almost always a range of possible values, rather than a single value. You can see the range easily by expressing it visually on a number line.

The shaded portion of the number line above shows the set of all numbers between \(-4\) and 0 excluding the end points \(-4\) and 0; this range would be expressed algebraically by the inequality \(-4 < x < 0\).
The shaded portion of the number line above shows the set of all numbers greater than $-1$, up to and including $3$; this range would be expressed algebraically by the inequality $-1 < x \leq 3$.

The other difference when solving an inequality—and the only thing you really have to remember—is that if you multiply or divide the inequality by a negative number, you have to reverse the direction of the inequality. For example, when you multiply both sides of the inequality $-3x < 2$ by $-1$, you get $3x > -2$.

**Example:**

Solve for $x$: $3 - \frac{x}{4} \geq 2$

Multiply both sides of the inequality by $4$: $12 - x \geq 8$

Subtract 12 from both sides: $-x \geq -4$

Multiply (or divide) both sides by $-1$ and change the direction of the inequality sign: $x \leq 4$

As you can see from the number line, the range of values that satisfies this inequality includes $4$ and all numbers less than $4$.

**Solving for One Unknown in Terms of Another**

In general, in order to solve for the value of an unknown, you need as many distinct equations as you have variables. If there are two variables, for instance, you need two distinct equations.
However, some GMAT problems do not require you to solve for the numerical value of an unknown. Instead you are asked to solve for one variable in terms of the other(s). To do so, isolate the desired variable on one side of the equation and move all the constants and other variables to the other side.

Example:

In the formula \( z = \frac{xy}{a + yb} \), solve for \( y \) in terms of \( x, z, a, \) and \( b \).

Clear the denominator by multiplying both sides by \( a + yb \).

\( (a + yb)z = xy \)

Remove parentheses by distributing. \( az + ybz = xy \)

Put all terms containing \( y \) on one side and all other terms on the other side. \( az = xy - ybz \)

Factor out the common factor, \( y \). \( az = y(x - bz) \)

Divide by the coefficient of \( y \) to get \( y \) alone. \( \frac{az}{x - bz} = y \)

Simultaneous Equations

We've already discovered that you need as many different equations as you have variables to solve for the actual value of a variable. When a single equation contains more than one variable, you can only solve for one variable in terms of the others.

This has important implications for Data Sufficiency. For sufficiency, you must have at least as many equations as you have variables.

On the GMAT, you will often have to solve two simultaneous equations, that is, equations that give you different information about the same two variables. There are two methods for solving simultaneous equations.
Method 1—Substitution

Step 1: Solve one equation for one variable in terms of the second.

Step 2: Substitute the result back into the other equation and solve.

Example:
If \( x - 15 = 2y \) and \( 6y + 2x = -10 \), what is the value of \( y \)?

Solve the first equation for \( x \) by adding 15 to both sides.

\[ x = 2y + 15 \]

Substitute \( 2y + 15 \) for \( x \) in the second equation:

\[ 6y + 2(2y + 15) = -10 \]
\[ 6y + 4y + 30 = -10 \]
\[ 10y = -40 \]
\[ y = -4 \]

Method 2—Adding to Cancel

Combine the equations in such a way that one of the variables cancels out. To solve the two equations \( 4x + 3y = 8 \) and \( x + y = 3 \), multiply both sides of the second equation by \( -3 \) to get: \( -3x - 3y = -9 \). Now add the two equations; the \( 3y \) and the \( -3y \) cancel out, leaving: \( x = -1 \).

Before you use either method, make sure you really do have two distinct equations. For example, \( 2x + 3y = 8 \) and \( 4x + 6y = 16 \) are really the same equation in different forms; multiply the first equation by 2, and you'll get the second.

Whichever method you use, you can check the result by plugging both values back into both equations and making sure they fit.
Example:

If \( m = 4n + 2 \), and \( 3m + 2n = 16 \), find the values of \( m \) and \( n \).

Since the first equation already expresses \( m \) in terms of \( n \), this problem is best approached by substitution.

Substitute \( 4n + 2 \) for \( m \) into \( 3m + 2n = 16 \), and solve for \( n \).

\[
3(4n + 2) + 2n = 16
\]
\[
12n + 6 + 2n = 16
\]
\[
14n = 10
\]
\[
n = \frac{5}{7}
\]

Now solve either equation for \( m \) by plugging in for \( n \).

\[
m = 4n + 2
\]
\[
m = 4\left(\frac{5}{7}\right) + 2
\]
\[
m = \frac{20}{7} + 2
\]
\[
m = \frac{20}{7} + \frac{14}{7}
\]
\[
m = \frac{34}{7}
\]

So \( m = \frac{34}{7} \) and \( n = \frac{5}{7} \).

Example:

If \( 3x + 3y = 18 \) and \( x - y = 10 \), find the values of \( x \) and \( y \).
You could solve this problem by the substitution method. But look what happens if you multiply the second equation by 3 and add it to the first:

\[
3x + 3y = 18 \\
+ (3x - 3y = 30) \\
\hline
6x = 48
\]

If \(6x = 48\), then \(x = 8\). Now you can just plug 8 into either equation in place of \(x\) and solve for \(y\). Your calculations will be simpler if you use the second equation: \(8 - y = 10\); \(-y = 2\); \(y = -2\)

**Simultaneous Equations in Data Sufficiency**

Data Sufficiency questions will sometimes test your understanding of how many equations you need to solve for a variable.

**Example:**

What is the value of \(x\)?

(1) \(x - 6y = 24\)

(2) \(4x + 2y = 16\)

Neither statement alone is sufficient, since each equation allows you only to solve for \(x\) in terms of another variable. However, both statements together give two different equations with two unknowns—enough information to find the value of \(x\). The answer is (C).

**Example:**

What is the value of \(x + y\)?

(1) \(x + 4y = -12\)

(2) \(5x + 5y = 18\)
We don’t need the value of either variable by itself, but their sum. The second statement gives us enough information. If we divided both sides by 5, we could find the value of \( x + y \). The answer is (B).

**Symbolism**

Don’t panic if you see strange symbols like *, ◊, and ♦ on a GMAT problem.

Problems of this type usually require nothing more than substitution. Read the question stem carefully for a definition of the symbols and for any examples of how to use them. Then, just follow the given model, substituting the numbers that are in the question stem.

**Example:**

An operation symbolized by \( \star \) is defined by the equation \( x \star y = x - \frac{1}{y} \). What is the value of \( 2 \star 7 \)?

The \( \star \) symbol is defined as a two-stage operation performed on two quantities, which are symbolized in the equation as \( x \) and \( y \). The two steps are: (1) find the reciprocal of the second quantity, and (2) subtract the reciprocal from the first quantity. To find the value of \( 2 \star 7 \), substitute the numbers 2 and 7 into the equation, replacing the \( x \) (the first quantity given in the equation) with the 2 (the first number given) and the \( y \) (the second quantity given in the equation) with the 7 (the second number given). The reciprocal of 7 is \( \frac{1}{7} \), and subtracting \( \frac{1}{7} \) from 2 gives you:

\[
2 - \frac{1}{7} = \frac{14}{7} - \frac{1}{7} = \frac{13}{7}
\]
When a symbolism problem only involves one quantity, the operations are usually a little more complicated. Nonetheless, you can follow the same steps to find the correct answer.

**Example:**

Let \( x^* \) be defined by the equation: \( x^* = \frac{-x^2}{1 - x^2} \). Evaluate \( \left(\frac{1}{2}\right)^* \).

\[
\left(\frac{1}{2}\right)^* = \frac{\left(\frac{1}{2}\right)^2}{1 - \left(\frac{1}{2}\right)^2} = \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{4} \times \frac{4}{3} = \frac{1}{3}
\]

Every once in a while, you'll see a symbolism problem that doesn't even include an equation. The definitions in this type of problem usually test your understanding of number properties.

**Example:**

\( \diamond x \) is defined as the largest even number that is less than the negative square root of \( x \). What is the value of \( \diamond 81 \)?

(A) –82  
(B) –80  
(C) –10  
(D) –8  
(E) 8

Plug in 81 for \( x \), and work backward logically. The negative square root of 81 is \( -9 \) because, \((-9)(-9) = 81\). The largest even number that is less than \( -9 \) is \( -10 \). (The number part of \( -8 \) is smaller than the number part of \( -9 \); however, you're dealing with negative numbers, so you have to look for the even number that would be just to the left of \( -9 \) along the number line.) Thus, the correct answer choice is (C), \(-10\).
Sequences

Sequences are lists of numbers. The value of a number in a sequence is related to its position in the list. Sequences are often represented on the GMAT as follows:

\[ s_1, s_2, s_3, \ldots, s_n, \ldots \]

The subscript part of each number gives you the position of each element in the series. \( s_1 \) is the first number in the list, \( s_2 \) is the second number in the list, and so on.

You will be given a formula that defines each element. For example, if you are told that \( s_n = 2n + 1 \), then the sequence would be \((2 \times 1) + 1, (2 \times 2) + 1, (2 \times 3) + 1, \ldots\), or 3, 5, 7, \ldots
Polynomials and Quadratics

The FOIL Method

When two binomials are multiplied, each term is multiplied by each term in the other binomial. This process is often called the FOIL method, because it involves adding the products of the First, Outer, Inner, and Last terms. Using the FOIL method to multiply out \((x + 5)(x - 2)\), the product of the first terms is \(x^2\), the product of the outer terms is \(-2x\), the product of the inner terms is \(5x\), and the product of the last terms is \(-10\). Adding, the answer is \(x^2 + 3x - 10\).

Factoring the Product of Binomials

Many of the polynomials that you’ll see on the GMAT can be factored into a product of two binomials by using the FOIL method backwards.

Example:

Factor the polynomial \(x^2 - 3x + 2\).

You can factor this into two binomials, each containing an \(x\) term. Start by writing down what you know:

\[ x^2 - 3x + 2 = (x \quad )(x \quad ) \]

You’ll need to fill in the missing term in each binomial factor. The product of the two missing terms will be the last term in the original polynomial: 2. The sum of the two missing terms will be the coefficient of the second term of the polynomial: \(-3\). Find the factors of 2 that add up to \(-3\). Since \((-1) + (-2) = -3\), you can fill the empty spaces with \(-1\) and \(-2\).

Thus, \(x^2 - 3x + 2 = (x - 1)(x - 2)\).
Note: Whenever you factor a polynomial, you can check your answer by using FOIL to multiply the factors and obtain the original polynomial.

**Factoring the Difference of Two Squares**

A common factorable expression on the GMAT is the difference of two squares (for example, $a^2 - b^2$). Once you recognize a polynomial as the difference of two squares, you'll be able to factor it automatically, since any polynomial of the form $a^2 - b^2$ can be factored into a product of the form $(a + b)(a - b)$.

**Example:**

Factor the expression $9x^2 - 1$.

$9x^2 = (3x)^2$ and $1 = 1^2$, so $9x^2 - 1$ is the difference of two squares. Therefore, $9x^2 - 1 = (3x + 1)(3x - 1)$.

**Factoring Polynomials of the Form $a^2 + 2ab + b^2$**

Any polynomial of this form is the square of a binomial expression, as you can see by using the FOIL method to multiply $(a + b)(a + b)$ or $(a - b)(a - b)$.

To factor a polynomial of this form, check the sign in front of the $2ab$ term. If it’s a plus sign, the polynomial is equal to $(a + b)^2$. If it’s a minus sign, the polynomial is equal to $(a - b)^2$.

**Example:**

Factor the polynomial $x^2 + 6x + 9$.

$x^2$ and 9 are both perfect squares, and $6x$ is $2(3x)$, which is twice the product of $x$ and 3, so this polynomial is of the form $a^2 + 2ab + b^2$ with $a = x$ and $b = 3$. Since there is a plus sign in front of the $6x$, $x^2 + 6x + 9 = (x + 3)^2$. 
Quadratic Equations

A quadratic equation is an equation of the form $ax^2 + bx + c = 0$. Many quadratic equations have two solutions. In other words, the equation will be true for two different values of $x$.

When you see a quadratic equation on the GMAT, you’ll generally be able to solve it by factoring the algebraic expression, setting each of the factors equal to zero, and solving the resulting equations.

Example:

$x^2 - 3x + 2 = 0$. Solve for $x$.

To find the solutions, or roots, start by factoring $x^2 - 3x + 2 = 0$ into $(x - 2)(x - 1) = 0$.

The product of two quantities equals zero only if one (or both) of the quantities equals zero. So if you set each of the factors equal to zero, you will be able to solve the resulting equations for the solutions of the original quadratic equation. Setting the two binomials equal to zero gives you:

$x - 2 = 0$ or $x - 1 = 0$

That means that $x$ can equal 2 or 1. As a check, you can plug each of those values in turn into $x^2 - 3x + 2 = 0$, and you’ll see that either value makes the equation work.
Alternative Strategies for Multiple-Choice Algebra

Backsolving

On GMAT Problem Solving questions, you may find it easier to attack algebra problems by backsolving.

To backsolve, substitute each answer choice into the equation until you find the one that satisfies the equation.

Example:

If $x^2 + 10x + 25 = 0$, what is the value of $x$?

(A) 25  
(B) 10  
(C) 5  
(D) $-5$  
(E) $-10$

The textbook approach to solve this problem would be to recognize the polynomial expression as the square of the binomial $(x + 5)$ and set $x + 5 = 0$. That’s the fastest way to arrive at the correct answer of $-5$.

But you could also plug each answer choice into the equation until you found the one that makes the equation true. Backsolving can be pretty quick if the correct answer is the first choice you plug in, but here, you have to get all the way down to choice (D) before you find that $(-5)^2 + 10(-5) + 25 = 0$. 


Example:

If \( \frac{5x}{3} + 9 = \frac{x}{6} + 18 \), \( x = \)

(A) 12  
(B) 8  
(C) 6  
(D) 5  
(E) 4

To avoid having to try all five answer choices, look at the equation and decide which choice(s), if plugged in for \( x \), would make your calculations easiest. Since \( x \) is in the numerators of the two fractions in this equation, and the denominators are 3 and 6, try plugging in a choice that is divisible by both 3 and 6. Choices (A) and (C) are divisible by both numbers, so start with one of them.

Choice (A):

\[
20 + 9 = 2 + 18 \\
29 \neq 20
\]

This is not true, so \( x \) cannot equal 12.

Choice (C):

\[
10 + 9 = 1 + 18 \\
19 = 19
\]

This is correct, so \( x \) must equal 6. Therefore, choice (C) is correct.

Backsolving may not be the fastest method for a multiple-choice algebra problem, but it’s useful if you don’t think you’ll be able to solve the problem in the conventional way.
Picking Numbers

On other types of multiple-choice algebra problems, especially where the answer choices consist of variables or algebraic expressions, you may want to pick numbers to make the problem less abstract. Evaluate the answer choices and the information in the question stem by picking a number and substituting it for the variable wherever the variable appears.

Example:

If \( a > 1 \), the ratio of \( 2a + 6 \) to \( a^2 + 2a - 3 \) is:

(A) \( 2a \)

(B) \( a + 3 \)

(C) \( \frac{2}{a - 1} \)

(D) \( \frac{2a}{3(3 - a)} \)

(E) \( \frac{a - 1}{2} \)

You can simplify the process by replacing the variable \( a \) with a number in each algebraic expression. Since \( a \) has to be greater than 1, why not pick 2? Then the expression \( 2a + 6 \) becomes \( 2(2) + 6 \), or 10. The expression \( a^2 + 2a - 3 \) becomes \( 2^2 + 2(2) - 3 = 4 + 4 - 3 = 5 \).

So now the question reads, “the ratio of 10 to 5 is what?” That’s easy enough to answer: 10:5 is the same as \( \frac{10}{5} \), or 2. Now you can just eliminate any answer choice that doesn’t give a result of 2 when you substitute 2 for \( a \). Choice (A) gives you 2(2), or 4, so discard it. Choice (B) results in 5—also not what you want. Choice (C) yields \( \frac{2}{1} \) or 2. That looks good, but you can’t stop here.
If another answer choice gives you a result of 2, you will have to pick another number for $a$ and reevaluate the expressions in the question stem and the choices that worked when you let $a = 2$.

Choice (D) gives you $\frac{2(2)}{3(3) - 2}$ or $\frac{4}{7}$, so eliminate choice (D).

Choice (E) gives you $\frac{2 - 1}{2}$ or $\frac{1}{2}$, so discard choice (E).

Fortunately, in this case, only choice (C) works out equal to 2, so it is the correct answer. But remember: when picking numbers, always check every answer choice to make sure you haven’t chosen a “special” number that works for more than one answer choice.

**Using Picking Numbers to Solve for One Unknown in Terms of Another**

It is also possible to solve for one unknown in terms of another by Picking Numbers. If the first number you pick doesn’t lead to a single correct answer, be prepared to either pick a new number (and spend more time on the problem) or settle for guessing strategically among the answers that you haven’t eliminated.
Example:

If \( \frac{x^2 - 16}{x^2 + 6x + 8} = y \) and \( x > -2 \), which of the following is an expression for \( x \) in terms of \( y \)?

(A) \( \frac{1 + y}{2 - y} \)

(B) \( \frac{2y + 4}{1 - y} \)

(C) \( \frac{4y - 4}{y + 1} \)

(D) \( \frac{2y - 4}{2 + y} \)

(E) \( \frac{y + 4}{y + 1} \)

Pick a value for \( x \) that will simplify your calculations. If you let \( x \) equal 4, then \( x^2 - 16 = 4^2 - 16 = 0 \), and so the entire fraction on the left side of the equation is equal to zero.

Now, substitute 0 for \( y \) in each answer choice in turn. Each choice is an expression for \( x \) in terms of \( y \), and since \( y = 0 \) when \( x = 4 \), the correct answer will have to give a value of 4 when \( y = 0 \). Just remember to evaluate all the answer choices, because you might find more than one that gives a result of 4.

Substituting 0 for \( y \) in choices (A), (C), and (D) yields \( \frac{1}{2}, \frac{4}{1}, \) and \( -\frac{4}{2} \), respectively, so none of those choices can be right.

But both (B) and (E) give results of 4 when you make the substitution; choosing between them will require picking another number.
Again, pick a number that will make calculations easy. If $x = 0$, then $y = \frac{x^2 - 16}{x^2 + 6x + 8} = \frac{0 - 16}{0 + 0 + 8} = \frac{-16}{8} = -2$

Therefore, $y = -2$ when $x = 0$. You don't have to try the new value of $y$ in all the answer choices, just in (B) and (E). When you substitute $-2$ for $y$ in choice (B), you get 0. That's what you're looking for but, again, you have to make sure it doesn't work in choice (E). Plugging $-2$ in for $y$ in (E) yields $-2$ for $x$, so (B) was correct.